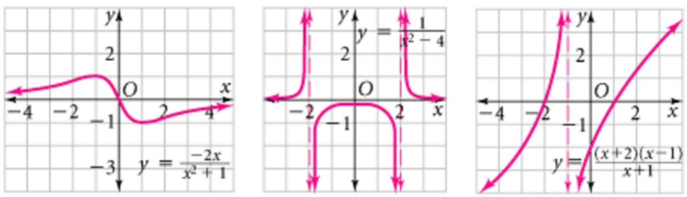
9.3 Rational functions and Their Graphs

An inverse variation is an example of a rational function. A rational function is a function that can be written as $f(x) = \frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomial functions and $Q(x) \neq 0$.

The domain of f(x) is all real numbers except where Q(x) = 0.

The graphs of the rational functions $y = \frac{-2x}{x^2+1}$, $y = \frac{1}{x^2-4}$, and $y = \frac{(x+2)(x-1)}{x+1}$ are shown below.



Point of Discontinuity: A point that is not on the graph because it will cause the denominator to be zero. The graph is discontinuous at x = a, where a is a real number which makes the denominator zero.

For each rational function, find any points of discontinuity.

$y = \frac{1}{x^2-16}$

$0 = x^2 - 16$

$16 = x^2$

$\pm\sqrt{16} = \sqrt{x^2}$

$\pm 4 = x$

$y = \frac{x^2-1}{x^2+3}$

$0 = x^2 + 3$

$-3 \qquad -3$

$\pm\sqrt{-3} = \sqrt{x^2}$

$\pm i\sqrt{3} = x$

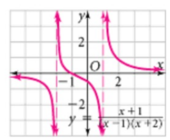
not real, continuous

$y = \frac{1}{x^2+2x-8}$

$0 = (x+4)(x-2)$

$x = -4 \quad x = 2$

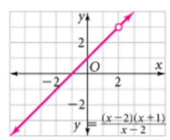
The graph of $y = \frac{x+1}{(x-1)(x+2)}$ is shown at the right. The zeros of the denominator are 1 and -2. The graph has vertical asymptotes at those points.



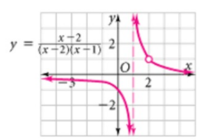
Graphing Calculator Hint

A hole in a graph is not displayed on a graphing calculator.

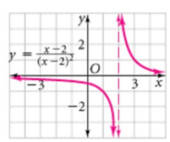
2 is a zero of both the numerator and the denominator of the rational function $y = \frac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}$. The graph of this function is the same as the graph of $y = x + 1$, except it has a hole at $x = 2$.



2 is a zero of both the numerator and the denominator of the rational function $y = \frac{\cancel{x-2}}{\cancel{(x-2)}(x-1)}$. The graph of this function is the same as the graph of $y = \frac{1}{x-1}$, except it has a hole at $x = 2$.



2 is a zero of both the numerator and the denominator of the rational function $y = \frac{x-2}{(x-2)^2}$. The graph of this function is exactly the same as the graph of $y = \frac{1}{x-2}$. The vertical asymptote is $x = 2$ and there is no hole.



Vertical Asymptotes:

The rational function $f(x) = \frac{P(x)}{Q(x)}$ has a point of discontinuity for each real zero of Q(x).

If P(x) and Q(x) have no common real zeros, then the graph of f(x) has a vertical asymptote at each real zero of Q(x).

If P(x) and Q(x) have a common real zero a, then there is a hole in the graph or a vertical asymptote at x = a.

Horizontal Asymptotes:

The graph of a rational function has at most one horizontal asymptote.

The graph of a rational function has a horizontal asymptote at $y = 0$ if the degree of the denominator is greater than the degree of the numerator.

If the degree of the numerator and the denominator are equal, then the graph has a horizontal asymptote at $y = \frac{a}{b}$, a is the coefficient of the term of highest degree in the numerator and b is the coefficient of the term of highest degree in the denominator.

If the degree of the numerator is greater than the degree of the denominator, then the graph has no horizontal asymptote.

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Describe the vertical and horizontal asymptotes and holes for the graph of each rational function.

$y = \frac{x-2}{(x-1)(x-3)}$

VA $x = 1$, $x = 3$

HA $y = 0$

no holes

zeros $(2,0)$

$y = \frac{\cancel{x-2}}{(\cancel{x-2})(x+3)}$ $\quad y = \frac{1}{x+3}$

VA $x = -3$

HA $y = 0$

hole $x = 2$

$y = \frac{-4x+3}{2x-1}$

VA $x = \frac{1}{2}$

HA $y = \frac{-4}{2} = -2$

no holes

zero $(\frac{3}{4}, 0)$

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Describe the vertical and horizontal asymptotes and holes fo the graph of each rational function.

$y = \frac{x^2-9}{3x^2+5} = \frac{(x+3)(x-3)}{3x^2+5}$

VA: none

HA: $y = \frac{1}{3}$

zeros $(-3,0)$, $(3,0)$

$y = \frac{x^2-1}{x+1} = \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$

simplifies to $y = x-1$

VA none

HA none

hole $x = -1$

zero $(1,0)$

$y = \frac{2x^2-3}{-5x^2+6}$

$0 = -5x^2+6$
$-6 \quad -6$
$\frac{-6}{-5} = \frac{-5x^2}{-5}$
$\pm\sqrt{\frac{6}{5}} = x^2$

VA $= x = \pm\sqrt{\frac{6}{5}}$

HA $y = -\frac{2}{5}$

zeros $(\sqrt{\frac{3}{2}}, 0)$ $(-\sqrt{\frac{3}{2}}, 0)$

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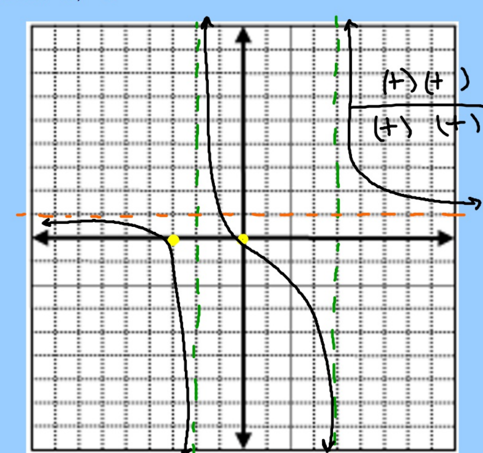
Use the horizontal and vertical asymptotes and zeros to graph.

Give a possible rule-zeros: 0, -3

VA: x = 4, x = -2

$y = \frac{x(x+3)}{(x-4)(x+2)}$



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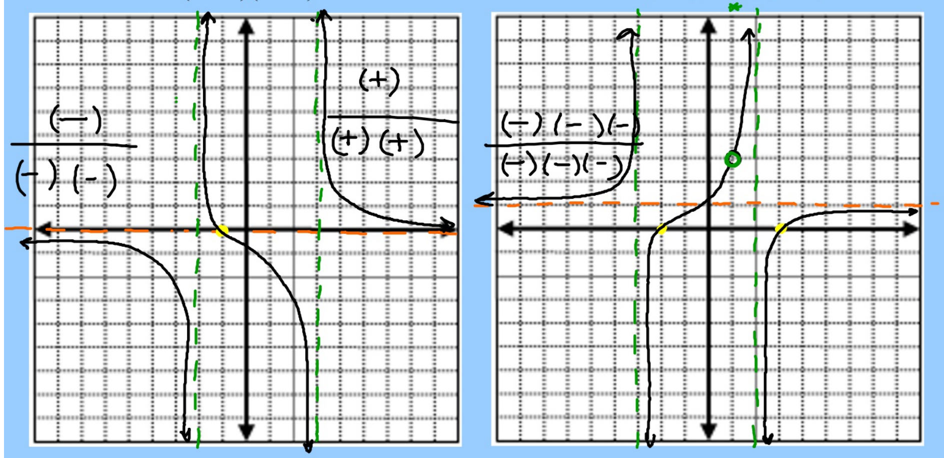
Use the horizontal and vertical asymptotes and zeros to graph.

$$y = \frac{x+1}{(x-3)(x+2)} \qquad y = \frac{\cancel{(x-1)}(x+2)(x-3)}{\cancel{(x-1)}(x-2)(x+3)}$$



homework:

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